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Approximate inverse systems which admit meshes

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Abstract

Recently, approximate inverse systems of spaces and their limits have been defined. These systems differ from usual inverse systems in that the bonding maps $p_{aa'}$ are not subject to the commutativity requirement $p_{aa'}p_{a'a''} = p_{aa''}$, $a \leqslant a' \leqslant a''$. Instead, the mappings $p_{aa'}p_{a'a''}$ and $p_{aa''}$ are allowed to differ up to a given normal covering $\mathscr{U}_a$ of X_a, called the *mesh* at $a \in A$. Imposing three conditions (A1)–(A3), one obtains a theory of *gauged* approximate systems, which has certain advantages over the usual theory of inverse systems. While conditions (A1) and (A3) depend on the meshes, (A2) does not. M. G. Charalambous initiated the study of approximate systems which satisfy only condition (A2) and therefore, makes no use of the meshes. The study of such systems was further pursued by the authors and by Vlasta Matijević. The present paper is devoted to the question, when does a system satisfying only condition (A2) admit meshes, which makes it a gauged system? A sufficient condition is found, which in some important cases becomes also a necessary condition.

Keywords: Inverse system; Approximate inverse system; Inverse limit; Resolution; Approximate resolution

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1. Introduction

A very useful and well-known method for constructing compact Hausdorff spaces X is to represent them as limits of inverse systems $X = (X_a, p_{aa'}, A)$ of compact polyhedra X_a, indexed by directed preordered sets $(A, \leqslant)$. Topologically complete spaces can be represented as limits of inverse systems of noncompact polyhedra. However, only systems called *resolutions* properly represent X and can

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be used to replace X, e.g., in shape theory (see [9]). These are systems, which satisfy two additional conditions (R1) and (R2) (see [3,4,9,12]).

It is well known that the limit X of an inverse system $\boldsymbol{X}$ of compact polyhedra X_a with $\dim X_a \leqslant n$ has (covering) dimension $\dim X \leqslant n$. However, the converse does not hold, i.e., $\dim X \leqslant n$ does not imply the existence of a system of compact polyhedra $\boldsymbol{X}$ with $\dim X_a \leqslant n$. This defect has been corrected by Mardešić and Rubin [6], who have defined *approximate systems* of metric compacta and have proved that a compact Hausdorff space X has $\dim X \leqslant n$ if and only if it is the limit of an approximate system of compact polyhedra X_a of dimension $\dim X_a \leqslant n$. A deeper application of approximate systems was made in [7], where it was proved that a compact Hausdorff space Y has integral cohomological dimension $\dim_{\mathbb{Z}} Y \leqslant n$ if and only if there exists a compact Hausdorff space X with $\dim X \leqslant n$ under a cell-like mapping $f: X \to Y$. The proof of the necessity required construction of a suitable X, which was obtained as the limit of an approximate system of compact polyhedra X_a of dimension $\dim X_a \leqslant n$.

Mardešić and Watanabe have put together the idea of resolution and the idea of an approximate system of compacta by introducing approximate systems and approximate resolutions of spaces [10]. It was then proved by Watanabe [15] that a topologically complete space X has covering dimension $\dim X \leqslant n$ if and only if it is the limit of an approximate resolution of polyhedra X_a of dimension $\dim X_a \leqslant n$.

According to [10], an *approximate resolution* is an approximate system of spaces $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ which satisfies conditions (R1) and (R2). Here the data A, X_a and $p_{aa'}$ are as in a usual inverse system (A is unbounded), but the structure is enriched by normal coverings $\mathscr{U}_a$ of X_a, $a \in A$, called *meshes* and subject to three conditions (A1)–(A3), which are natural analogues of the Mardešić–Rubin conditions. Before we state these conditions, let us agree that $\operatorname{Cov}(X)$ denotes the set of all normal coverings of X. For $\mathscr{U}, \mathscr{V} \in \operatorname{Cov}(X)$, $\mathscr{V} \prec \mathscr{U}$ means that $\mathscr{V}$ refines $\mathscr{U}$. If $\mathscr{V} \in \operatorname{Cov}(Y)$ and $f, f': X \to Y$ are mappings, $(f, f') \prec \mathscr{V}$ means that f and f' are $\mathscr{V}$-near mappings, i.e., for $x \in X$ there is a $V \in \mathscr{V}$ such that $f(x), f'(x) \in V$.

(A1) $(\forall a_2 \geqslant a_1 \geqslant a)$

$$(p_{aa_1} p_{a_1a_2}, p_{aa_2}) \prec \mathscr{U}_a.$$

(A2) $(\forall a \in A)(\forall \mathscr{U} \in \operatorname{Cov}(X_a))(\exists a' \geqslant a)(\forall a_2 \geqslant a_1 \geqslant a')$

$$(p_{aa_1} p_{a_1a_2}, p_{aa_2}) \prec \mathscr{U}.$$

(A3) $(\forall a \in A)(\forall \mathscr{U} \in \operatorname{Cov}(X_a))(\exists a' \geqslant a)(\forall a'' \geqslant a')$

$$\mathscr{U}_{a''} \prec p_{aa''}^{-1}(\mathscr{U}).$$

Noticing that condition (A2) does not use meshes, Charalambous [1] has considered approximate systems (of uniform spaces) $\boldsymbol{X} = (X_a, p_{aa'}, A)$ with no meshes and subject only to condition (A2). Further study of these systems (of topological spaces) was performed by the authors and Matijević [11]. We use the term *approximate system* for systems with no meshes satisfying condition (A2) and we denote such systems by boldface characters. We refer to systems with meshes as to *gauged approximate systems* and denote them by script characters. The purpose of the present paper is to clarify the relationship between these two notions. Our main concern is the question, when does an approximate system $\mathbf{X}$ admit meshes such that condition (A3) is satisfied? This condition was first introduced by Watanabe [14] in the case of usual (commutative) inverse systems and the question makes sense in that case as well. Our main results are Theorems 2.8 and 3.2.

2. A sufficient condition for gauging a system

A *basis* of (open) normal coverings of a space X is a collection $\mathscr{C}$ of normal coverings such that every normal covering $\mathscr{U}$ of X admits a refinement $\mathscr{V} \in \mathscr{C}$. We denote by $\mathrm{cw}(X)$ (*covering weight*) the minimal cardinal of a basis of normal coverings of X. As usual in the theory of cardinal invariants, we put $\mathrm{cw}(X) = \aleph_0$ if this minimal cardinal is finite. In a Tychonoff space X normal coverings form the maximal uniform structure compatible with the topology of X and $\mathrm{cw}(X)$ is the uniform weight $u(X)$ of the corresponding uniform space X. Recall that in paracompact spaces every open covering is normal and in normal spaces every finite open covering is normal.

Example 2.1. Let X be a discrete space of infinite cardinality α. Then the weight $w(X) = \alpha$ and $\mathrm{cw}(X) = \aleph_0$. Therefore, if $\alpha > \aleph_0$, then $w(X) > \mathrm{cw}(X)$.

Example 2.2. If X is a compact Hausdorff space, then $\mathrm{cw}(X)$ equals the weight $w(X)$. Indeed, if $\mathscr{B}$ is a basis of open sets of X of cardinality $w(X)$, then finite open coverings formed by the members of $\mathscr{B}$ form a basis of open coverings of cardinality $w(X)$, which shows that $\mathrm{cw}(X) \leqslant w(X)$. Conversely, if $\mathscr{C}$ is a basis of open coverings of cardinality $\mathrm{cw}(X)$, then one can assume that $\mathscr{C}$ consists of finite coverings only. Then the set $\mathscr{B}$ of all members V of all coverings $\mathscr{V} \in \mathscr{C}$ is also of cardinality $\mathrm{cw}(X)$. Considering coverings of the form $\mathscr{U} = (U, X \setminus \{x\})$, where U is an open neighborhood of $x \in X$, it is easy to see that $\mathscr{B}$ is a basis for the topology of X and therefore, $w(X) \leqslant \mathrm{cw}(X)$.

Example 2.3. Let S be a discrete space of infinite cardinality α and let $X = S \times I$, where $I = [0, 1]$. No collection $\mathscr{C}$ of α open coverings of X can be a basis of open coverings of X and thus $\mathrm{cw}(X) > \alpha$. In order to prove the assertion assume that the members $\mathscr{U}_s$ of $\mathscr{C}$ are indexed by $s \in S$. To each $s \in S$ assign a member $U_s \in \mathscr{U}_s$, which contains the point $(s, 1)$. Then choose an open neighborhood V_s of this point so small that it does not contain U_s and it misses all the sets $s' \times I$, where $s' \neq s$. Let $\mathscr{V}$ be the open covering of X, which consists of all the open sets V_s,

$s \in S$, and of the open set $V = X \setminus \bigcup\{(s, 1) \mid s \in S\}$. The member U_s of $\mathscr{U}_s$ is neither contained in V_s nor in any of the remaining members of $\mathscr{V}$, because none of them contains the point $(s, 1)$. Therefore, $\mathscr{U}_s$ does not refine $\mathscr{V}$.

Example 2.4. Note that, for a closed subset X' of a paracompact space X, one has $\mathrm{cw}(X') \leqslant \mathrm{cw}(X)$. The real line $\mathbb{R}$ contains a closed subset homeomorphic to $\mathbb{N} \times I$. Consequently, by Example 2.3, $\mathrm{cw}(\mathbb{R}) \geqslant \mathrm{cw}(\mathbb{N} \times I) > \aleph_0 = w(\mathbb{R})$.

Example 2.5. Let A be an infinite directed ordered set and let X be a space such that $\mathrm{cw}(X) > \mathrm{card}(A)$. Let $\boldsymbol{X} = (X_a, p_{aa'}, A)$, where $X_a = X$ for all $a \in A$ and $p_{aa'} = \mathrm{id}$ for $a \leqslant a'$. Then $\boldsymbol{X}$ is an example of an inverse system (approximate system), which does not admit meshes satisfying condition (A3). Assume the contrary, i.e., that there exist coverings $\mathscr{U}_a \in \mathrm{Cov}(X_a)$, which satisfy (A3). Consider any $\mathscr{U} \in \mathrm{Cov}(X)$. Then choose any $a \in A$ and apply (A3) to a and $\mathscr{U}$. One obtains an $a' \geqslant a$ such that $\mathscr{U}_{a'} \prec \mathscr{U}$. This shows that $\mathscr{C} = \{\mathscr{U}_{a'} \mid a' \geqslant a\}$ is a basis of normal coverings of X whose cardinality is $\leqslant \mathrm{card}(A)$. Consequently, $\mathrm{cw}(X) \leqslant \mathrm{card}(A)$, which contradicts the assumption that $\mathrm{cw}(X) > \mathrm{card}(A)$. In view of Examples 2.2 and 2.4, we obtain specific examples of approximate systems, which do not allow gauging, by putting $A = \mathbb{N}$ and $X = I^\tau$, where τ is an uncountable cardinal, or by taking $X = \mathbb{R}$.

Example 2.6. Let X be a normal space, which is not metrizable. Let $\boldsymbol{X}$ be the inverse system, indexed by $\mathbb{N}$ and consisting of copies of X and of identity mappings. Then $\boldsymbol{X}$ does not admit meshes satisfying condition (A3). Assume that such meshes $\mathscr{U}_n$, $n \in \mathbb{N}$, exist. Because of Moore's metrization theorem (see [2, 5.4.2]), a contradiction will be obtained if we show that $(\mathscr{U}_1, \mathscr{U}_2, \ldots)$ is a strong development of X, i.e., for any $x \in X$ and any open neighborhood U of x, there exist an open set V and an n such that

$$x \in V \subseteq \mathrm{st}(V, \mathscr{U}_n) \subseteq U.$$

For given x and U, choose an open neighborhood V of x such that $\mathrm{Cl}(V) \subseteq U$. Then $\mathscr{V} = \{U, X \setminus \mathrm{Cl}(V)\}$ is a normal open covering of $X = X_1$. By (A3), there exists an $n \in \mathbb{N}$ such that $\mathscr{U}_n \prec \mathscr{V}$. If $W \in \mathscr{U}_n$ and $W \cap V \neq \emptyset$, then W cannot be contained in $X \setminus \mathrm{Cl}(V)$ and therefore, $W \subseteq U$. This shows that $\mathrm{st}(V, \mathscr{U}_n) \subseteq U$, i.e., $(\mathscr{U}_1, \mathscr{U}_2, \ldots)$ is indeed a strong development of X.

Remark 2.7. In [5,11] it is shown that with every approximate system $\boldsymbol{X}$ one can associate a gauged approximate system $\mathscr{X}$, which uses only spaces and mappings from $\boldsymbol{X}$ and has the same limit as $\boldsymbol{X}$. Moreover, if $\boldsymbol{X}$ is an approximate resolution of X, then so is $\mathscr{X}$. Furthermore, $\mathscr{X}$ is unique up to isomorphism in the category **APRES** of gauged cofinite approximate resolutions of topologically complete spaces (defined in [10]), provided $\mathscr{X}$ is a cofinite approximate resolution of topologically complete spaces. However, the index set of $\mathscr{X}$ differs from the index set of X.

We now given a condition which makes the introduction of meshes in an approximate system possible.

Theorem 2.8. *Let $X=(X_a, p_{aa'}, A)$ be an approximate system over a cofinite index set A. If X satisfies condition*
(C) $\operatorname{card}(A) \geqslant \operatorname{cw}(X_a)$ $(\forall a \in A)$,
then there exist coverings $\mathscr{U}_a \in \operatorname{Cov}(X_a)$, $a \in A$, such that the gauged system $\mathscr{X}=(X_a, \mathscr{U}_a, p_{aa'}, A)$ (in addition to (A2)*) satisfies condition* (A3).

Condition (C) was first considered by Uglešić in connection with stability of approximate systems [13].

Proof. First notice that for each $a \in A$, the sets $A_a=\{a' \in A \mid a' \geqslant a\}$, $a \in A$, and A have the same cardinals. Indeed, since A is directed and cofinite, for each $a_0 \in A$, there exists an increasing function $\varphi: A \to A$ such that $\varphi(a) \geqslant a_0$, for each $a \in A$. Using the assumption that A is infinite, one can achieve that φ is strictly increasing and therefore an injection. Now (C) implies

$$\operatorname{card}(A_a) \geqslant \operatorname{cw}(X_a) \quad (\forall a \in A). \tag{2.1}$$

Therefore, for each $a \in A$, there exists a surjection $\psi_a: A_a \to \mathscr{C}(X_a)$, where $\mathscr{C}(X_a)$ is a basis of normal coverings of X_a of cardinality $\operatorname{cw}(X_a)$. Order the members of $\mathscr{C}(X_a)$ by putting $\mathscr{U} \leqslant \mathscr{V}$ if $\mathscr{V} \prec \mathscr{U}$. Using cofiniteness of A, one can produce an increasing function $\phi_a: A_a \to \mathscr{C}(X_a)$, such that $\phi_a \geqslant \psi_a$, i.e., if $a_1 \leqslant a_2$, then the covering $\phi_a(a_2)$ refines the covering $\phi_a(a_1)$. Moreover, for every covering $\mathscr{U} \in \operatorname{Cov}(X_a)$, there exists an $a' \geqslant a$ such that $\phi_a(a')$ refines $\mathscr{U}$. For $a \in A$, we now define a normal covering $\mathscr{U}_a \in \operatorname{Cov}(X_a)$ by

$$\mathscr{U}_a=\bigwedge_{i=1}^{n} p_{a_i a}^{-1}(\phi_{a_i}(a)), \tag{2.2}$$

where $\{a_1, \ldots, a_n\}$ are all the predecessors of a. In order to show that $\mathscr{X}=(X_a, \mathscr{U}_a, p_{aa'}, A)$ satisfies (A3), consider an $a \in A$ and a $\mathscr{U} \in \operatorname{Cov}(X_a)$. Since $\mathscr{C}(X_a)$ is a basis of normal coverings of X_a, there exists a $\mathscr{V} \in \mathscr{C}(X_a)$ such that $\mathscr{V} \prec \mathscr{U}$. By the surjectivity of ψ_a, there exists an $a' \geqslant a$, such that $\psi_a(a')=\mathscr{V}$. Consequently,

$$\phi_a(a') \prec \psi_a(a') \prec \mathscr{U}. \tag{2.3}$$

Now let $a'' \geqslant a'$. Then

$$\mathscr{U}_{a''} \prec p_{aa''}^{-1}(\phi_a(a'')), \tag{2.4}$$

because a is a predecessor of a'' and thus, $p_{aa''}^{-1}(\phi_a(a''))$ is one of the terms in the expression (2.2), which defines $\mathscr{U}_{a''}$. However,

$$\phi_a(a'') \prec \phi_a(a') \tag{2.5}$$

and therefore (2.3) implies

$$p_{aa''}^{-1}(\phi_a(a'')) \prec p_{aa''}^{-1}(\mathscr{U}). \tag{2.6}$$

(2.4) and (2.6) yield the desired conclusion

$$\mathscr{U}_{a''} \prec p_{aa''}^{-1}(\mathscr{U}). \quad \square \tag{2.7}$$

Corollary 2.9. *Let $X = (X_a, p_{aa'}, A)$ be an approximate system of metric compacta over a cofinite and unbounded index set A. Then there exist coverings $\mathscr{U}_a \in Cov(X_a)$, $a \in A$, such that the gauged system $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ (in addition to* (A2)) *satisfies condition* (A3).

Proof. For each $a \in A$, the weight $w(X_a) = \aleph_0$. By Example 2.1, $\mathrm{cw}(X_a) = w(X_a)$. Since A is infinite, condition (C) is fulfilled. In the case of commutative systems, the corollary is Watanabe's Proposition 3.8 of [14]. $\square$

Remark 2.10. Condition (A1), although convenient, is not essential because, if not valid, it can be introduced by weakening the ordering of the index set A. Indeed, let $A^* = (A, \leqslant^*)$ be equal to A as a set and let us put $a \leqslant^* a'$ provided $a = a'$ or $a < a'$ and whenever $a_2 \geqslant a_1 \geqslant a'$ one has

$$(p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leqslant \mathscr{U}_a.$$

Note that, because of (A2), each $a \in A$ admits an $a' >^* a$. Moreover, if $a <^* a'$ and $a' \leqslant a''$, then $a <^* a''$. Using these facts, it is easy to see that $\mathscr{X}^* = (X_a, \mathscr{U}_a, p_{aa'}, A^*)$ has properties (A1)–(A3) if $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ has properties (A2) and (A3) (see [8, Theorem 3.5]).

3. Necessity of condition (C)

In this section we consider approximate systems X for which (C) is also a necessary condition for the existence of meshes satisfying condition (A3). We first have the following theorem.

Theorem 3.1. *Let $X = (X_a, p_{aa'}, A)$ be an approximate system of compact Hausdorff spaces X_a with surjective bonding mappings $p_{aa'}$ and infinite index set A. If there exist open coverings $\mathscr{U}_a \in \mathrm{Cov}(X_a)$, $a \in A$, such that $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ satisfies* (A3), *then condition* (C) *holds, i.e.,* $\mathrm{cw}(X_a) = w(X_a) \leqslant \mathrm{card}(A)$, *for all* $a \in A$.

We also have a result for approximate systems X of arbitrary spaces. However, in this case we need an additional assumption on the Lindelöf numbers $l(X_a)$ (based on normal coverings). By definition, $l(X)$ is the minimal infinite cardinal λ such that every $\mathscr{U} \in \mathrm{Cov}(X)$ admits a refinement $\mathscr{V} \in \mathrm{Cov}(X)$ of cardinality $\leqslant \lambda$.

Theorem 3.2. *Let $X = (X_a, p_{aa'}, A)$ be an approximate system of topological spaces X_a, whose Lindelöf numbers $l(X_a)$, $a \in A$, satisfy the inequality*

(D) $2^{l(X_a)} \leqslant \mathrm{card}(A)$,

and let each $p_{aa'}(X_{a'})$ be a subset dense on X_a. If there exist coverings $\mathscr{U}_a \in \mathrm{Cov}(X_a)$, $a \in A$, such that $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ satisfies condition (A3), *then condition* (C) *holds, i.e.,* $\mathrm{cw}(X_a) \leqslant \mathrm{card}(A)$ *for all* $a \in A$.

Notice that for compact spaces X_a every covering admits a refinement of $k < \aleph_0$ members and $2^k < \aleph_0$, so that the analogue of (D) holds, whenever $\mathrm{card}(A) \geqslant \aleph_0$. Therefore, the proof of Theorem 3.1 is obtained from the proof of Theorem 3.2 by obvious changes and we omit it.

Proof of Theorem 3.2. For an arbitrary $a_0 \in A$ we shall exhibit a basis $\mathscr{C}$ of normal coverings of X_{a_0} such that

$$\mathrm{card}(\mathscr{C}) \leqslant \mathrm{card}(A) \tag{3.1}$$

and therefore $\mathrm{cw}(X_{a_0}) \leqslant \mathrm{card}(A)$.

Consider any $a \geqslant a_0$ and choose a normal covering $\mathscr{L}_a \in \mathrm{Cov}(X_a)$, which refines the mesh $\mathscr{U}_a$ and satisfies

$$\mathrm{card}(\mathscr{L}_a) \leqslant l(X_a). \tag{3.2}$$

Let $\mathscr{M}_a$ be the set of all subsets $\mathscr{M}$ of $\mathscr{L}_a$. Note that

$$\mathrm{card}(\mathscr{M}_a) = 2^{\mathrm{card}(\mathscr{L}_a)} \leqslant 2^{l(X_a)}. \tag{3.3}$$

Let $\mathscr{N}_a$ be the set of all subsets $\mathscr{N}$ of $\mathscr{M}_a$ of cardinality $\mathrm{card}(\mathscr{N}) \leqslant l(X_{a_0})$. Note that

$$\mathrm{card}(\mathscr{N}_a) \leqslant \left(\mathrm{card}(\mathscr{M}_a)\right)^{l(X_{a_0})} \leqslant \left(2^{l(X_a)}\right)^{l(X_{a_0})} = 2^{l(X_a)l(X_{a_0})} \leqslant \mathrm{card}(A), \tag{3.4}$$

because $l(X_a)l(X_{a_0}) = \max\{l(X_a), l(X_{a_0})\}$ and (D) applies.

With each $\mathscr{M} \in \mathscr{M}_a$ we associate an open subset $G(\mathscr{M})$ of X_{a_0}, defined by

$$G(\mathscr{M}) = \mathrm{Int}\ \mathrm{Cl}\left(\bigcup\left\{p_{a_0a}(U) \mid U \in \mathscr{M}\right\}\right). \tag{3.5}$$

Moreover, with each $\mathscr{N} \in \mathscr{N}_a$ we associate a collection $\mathscr{G}(\mathscr{N})$ of open subsets of X_{a_0}, defined by

$$\mathscr{G}(\mathscr{N}) = \{G(\mathscr{M}) \mid \mathscr{M} \in \mathscr{N}\}. \tag{3.6}$$

Let $\mathscr{C}_a$ be the collection of all families $\mathscr{G}(\mathscr{N})$, $\mathscr{N} \in \mathscr{N}_a$, which are normal coverings of X_{a_0}. Note that

$$\mathrm{card}(\mathscr{C}_a) \leqslant \mathrm{card}(\mathscr{N}_a) \leqslant \mathrm{card}(A). \tag{3.7}$$

Finally, let

$$\mathscr{C} = \bigcup\{\mathscr{C}_a \mid a \geqslant a_0\}. \tag{3.8}$$

Clearly, $\mathscr{C}$ is a collection of normal coverings of X_{a_0} such that

$$\mathrm{card}(\mathscr{C}) \leqslant (\mathrm{card}(A))^2 = \mathrm{card}(A). \tag{3.9}$$

It remains to prove that $\mathscr{C}$ is a basis of normal coverings of X_{a_0}.

Let $\mathscr{U}$ be a normal covering of X_{a_0}. We choose normal coverings $\mathscr{V}$, $\mathscr{W}$ of X_{a_0} such that $\mathscr{V}$ is a star-refinement of $\mathscr{U}$ and $\mathscr{W}$ is a star-refinement of $\mathscr{V}$. Moreover, we require that

$$\operatorname{card}(\mathscr{W}) \leqslant l(X_{a_0}). \tag{3.10}$$

By (A3), there exists an $a \geqslant a_0$ such that

$$p_{a_0 a}(\mathscr{U}_a) \leqslant \mathscr{W}. \tag{3.11}$$

Therefore, also

$$p_{a_0 a}(\mathscr{L}_a) \leqslant \mathscr{W}. \tag{3.12}$$

With each $W \in \mathscr{W}$, we associate a set $\mathscr{M}(W) \in \mathscr{M}_a$, defined by

$$\mathscr{M}(W) = \{L \in \mathscr{L}_a \mid p_{a_0 a}(L) \cap W \neq \emptyset\}. \tag{3.13}$$

Then we define a subset $\mathscr{N}(\mathscr{W})$ of $\mathscr{M}_a$ by

$$\mathscr{N}(\mathscr{W}) = \{\mathscr{M}(W) \mid W \in \mathscr{W}\}. \tag{3.14}$$

Note that (3.10) implies $\mathscr{N}(\mathscr{W}) \in \mathscr{N}_a$. Moreover, (3.6) and (3.14) show that

$$\mathscr{G}(\mathscr{N}(\mathscr{W})) = \{G(\mathscr{M}(W)) \mid W \in \mathscr{W}\}. \tag{3.15}$$

We will show that $\mathscr{G}(\mathscr{N}(\mathscr{W}))$ is a normal covering of X_{a_0} and therefore,

$$\mathscr{G}(\mathscr{N}(\mathscr{W})) \in \mathscr{C}_a \subseteq \mathscr{C}. \tag{3.16}$$

Let us first show that, for each $W \in \mathscr{W}$,

$$W \subseteq G(\mathscr{M}(W)). \tag{3.17}$$

For any $y \in W$, choose an open neighborhood H of y contained in W. Since $p_{a_0 a}(X_a)$ is dense on X_{a_0}, there exists a point $x \in X_a$ such that

$$p_{a_0 a}(x) \in H \subseteq W. \tag{3.18}$$

Since $\mathscr{L}_a$ covers X_a, there is an $L \in \mathscr{L}_a$ such that $x \in L$ and therefore, $p_{a_0 a}(x) \in p_{a_0 a}(L)$. Consequently, $p_{a_0 a}(L) \cap W \supseteq p_{a_0 a}(L) \cap H \neq \emptyset$, which shows that $L \in \mathscr{M}(W)$ and therefore

$$H \cap \left(\bigcup\{p_{a_0 a}(L) \mid L \in \mathscr{M}(W)\}\right) \neq \emptyset. \tag{3.19}$$

However, (3.19) implies

$$y \in \operatorname{Cl}\left(\bigcup\{p_{a_0 a}(L) \mid L \in \mathscr{M}(W)\}\right), \tag{3.20}$$

for each $y \in W$. Since W is open, (3.20) and (3.5) prove (3.17).

$\mathscr{W}$ is a covering of X_{a_0} and therefore, (3.15) and (3.17) show that $\mathscr{G}(\mathscr{N}(\mathscr{W}))$ is a covering of X_{a_0}, refined by $\mathscr{W}$. Since $\mathscr{W}$ is normal, so must be $\mathscr{G}(\mathscr{N}(\mathscr{W}))$. This proves that $\mathscr{G}(\mathscr{N}(\mathscr{W}))$ belongs to $\mathscr{C}_a$.

It remains to show that $\mathscr{G}(\mathscr{N}(\mathscr{W}))$ refines $\mathscr{U}$. Because of (3.15), we must show that each $W \in \mathscr{W}$ admits a $U \in \mathscr{U}$ such that

$$G(\mathscr{M}(W)) \subseteq U. \tag{3.21}$$

If $L \in \mathscr{M}(W)$, (3.13) implies $p_{a_0a}(L) \cap H \neq \emptyset$. On the other hand, by (3.12), there exists an $W' \in \mathscr{W}$ such that

$$p_{a_0a}(L) \subseteq W'. \tag{3.22}$$

Therefore, $W' \cap W \neq \emptyset$ and thus

$$p_{a_0a}(L) \subseteq W' \subseteq \mathrm{St}(W, \mathscr{W}). \tag{3.23}$$

Using (3.5), we now conclude that

$$G(\mathscr{M}(W)) \subseteq \mathrm{Cl}\big(\bigcup\{p_{a_0a}(L) \mid L \in \mathscr{M}(W)\}\big) \subseteq \mathrm{Cl}(\mathrm{st}(W, \mathscr{W})). \tag{3.24}$$

Since $\mathscr{W}$ is a star-refinement of $\mathscr{V}$, there is a $V \in \mathscr{V}$ such that $\mathrm{St}(W, \mathscr{W}) \subseteq V$ and therefore, $G(\mathscr{M}(W)) \subseteq \mathrm{Cl}(V)$. Finally, since $\mathscr{V}$ is a star-refinement of $\mathscr{U}$, there is a $U \in \mathscr{U}$ such that $\mathrm{Cl}(V) \subseteq U$ and therefore, (3.21) holds. □

Definition 3.3. A gauged approximate resolution $\mathscr{X} = (X_a, \mathscr{U}_a, p_{aa'}, A)$ from **APRES** is said to be *stable* provided for each pair of indexes $a \leqslant a'$, there exists a normal open covering $\mathscr{U}_{aa'} \in \mathrm{Cov}(X_a)$ such that for any choice of mappings $p'_{aa'}: X_{a'} \to X_a$ satisfying the condition

$$(p_{aa'}, p'_{aa'}) \prec \mathscr{U}_{aa'}, \tag{3.25}$$

$\mathscr{X}' = (X_a, \mathscr{U}_a, p'_{aa'}, A)$ is an approximate resolution isomorphic to $\mathscr{X}$ in **APRES**.

It has been proved by Uglešić [13] that condition (C) implies stability. Therefore, Theorems 3.1 and 3.2 have the following consequences.

Corollary 3.4. *Every gauged approximate system, consisting of compact Hausdorff spaces and surjective bonding mappings, is stable.*

Corollary 3.5. *Every gauged approximate resolution from* **APRES**, *which satisfies condition* (D), *and whose bonding mappings p_{a_0a} map X_a densely on X_{a_0}, is stable.*

Example 3.6. The following example shows that in Theorem 3.1 one cannot omit the surjectivity assumption. Let $\mathscr{X} = (X_n, \mathscr{U}_n, p_{nn'}, \mathbb{N})$, where X_1 is a nonmetrizable compact Hausdorff space and $X_n = \{x_n\}$ is a singleton for $n > 1$. Furthermore, let $\mathscr{U}_n = \{X_n\}$ for $n \in \mathbb{N}$ and $p_{nn'}(x_{n'}) = x_n$. Then $\mathscr{X}$ satisfies all three conditions (A1)–(A3), but not (C) because $\mathrm{cw}(X_1) = w(X_1) > \aleph_0 = \mathrm{card}(\mathbb{N})$.

Example 3.7. The next example shows that in Theorem 3.2, even in the case when all X_a are locally compact metric spaces, one cannot omit the assumption (D). Let α be an infinite cardinal. We will define a gauged inverse sequence $\mathscr{X} = (X_n, \mathscr{U}_n, p_{nn'}, \mathbb{N})$, which satisfies condition (A3), all X_n are locally compact metric spaces and $\mathrm{cw}(X_n) > \alpha$. Therefore, condition (C) does not hold.

For any space Y let Y^* denote the space obtained from Y by replacing its topology with the discrete topology. Choose a locally compact metric space X such that $\mathrm{cw}(X) > \alpha$ (see Example 2.3). We define the spaces X_n by induction on n putting $X_1 = X$ and $X_{n+1} = (X_n)^* \times X$. The bonding map $p_{nn+1}: X_{n+1} \to X_n$ is defined as the composition of the projection $(X_n)^* \times X \to (X_n)^*$ and the identity mapping $(X_n)^* \to X_n$. Since both of these mappings are continuous and surjective, so is p_{nn+1}. For $k \geqslant 0$, we put $p_{n,n+k+1} = p_{nn+1} \cdots p_{n+k,n+k+1}$. Note that $p_{n,n+k+1}$ maps the set $x \times X$, $x \in (X_{n+k})^*$, to the single point $p_{nn+1} \ldots p_{n+k-1,n+k}(x)$. We define the meshes $\mathscr{U}_n$ by $\mathscr{U}_1 = \{X\}$, $\mathscr{U}_{n+1} = \{x \times X \mid x \in X_n\}$. Since $(X_n)^*$ is a discrete space, $\mathscr{U}_{n+1}$ is indeed an open covering of X_{n+1}.

We claim that for any $n \in \mathbb{N}$ and $\mathscr{U} \in \mathrm{Cov}(X_n)$, $n' = n + 1$ satisfies (A3). Indeed, if $n'' \geqslant n'$, i.e., $n'' = n + k + 1$, where $k \geqslant 0$, $p_{nn''}$ maps the members of $\mathscr{U}_{n''} = \{x \times X \mid x \in X_{n+k}\}$ to single points and therefore, $p_{nn''}(\mathscr{U}_{n''})$ refines $\mathscr{U}$. It remains to show that $\mathrm{cw}(X_{n+1}) > \alpha$. Consider any basis $\mathscr{C}$ of open coverings of X_{n+1}. Let $\mathscr{C}'$ consist of all the restrictions of the members of $\mathscr{C}$ to the subset $x_0 \times X$, where x_0 is a fixed element of X_n. Since $x_0 \times X$ is closed in X_{n+1}, $\mathscr{C}'$ is a basis of open coverings of X and therefore, $\mathrm{card}(\mathscr{C}) \geqslant \mathrm{card}(\mathscr{C}') \geqslant \mathrm{cw}(X) > \alpha$. This proves that indeed $\mathrm{cw}(X_{n+1}) > \alpha$.

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